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A note on the barotropic instability of the Bickley jet

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The linear stability of the Bickley jet in the framework of the beta-plane approximation is considered, with the objective of presenting analytical calculations which add to previous numerical computations. It is well-known that the equation governing the neutral solutions which are analytic at the critical layer can be transformed into an associated Legendre equation. It turns out that this particular equation has simple closed-form solutions other than those known already, which are the Legendre polynomial of degree two, and two associated Legendre functions of the first kind, respectively. This observation makes it possible to find analytic neutral modes and corresponding neutral curves in the (β, k)-plane not known previously, both for the bounded and the unbounded Bickley jet. Here β denotes the beta-parameter and k the wavenumber. These neutral curves comprise parts of the stability boundary. It is shown that the line segment $(\beta = -2,\ 0 < k < \sqrt{2})$ is part of the stability boundary for the unbounded Bickley jet, so the region for the unstable radiating modes is larger than the one obtained previously. Also, analytic sinuous and varicose modes and corresponding neutral curves are found in the case of the bounded flow where only numerical calculations have previously been presented. Furthermore, local stability analysis reveals weakly amplified modes with wave speed outside the range of the velocity profile for the Bickley jet. This is rather rare, although Pedlosky's theorem allows for it, and there are only a few examples of flows in which such modes occur. Here these modes are sinuous modes and occur when the flow is both bounded and unbounded.

1. Introduction

Studies of the barotropic instability of two-dimentional jets, which play an important role in many geophysical and astrophysical flows, have been the objective of several previous papers, and many of them (Lipps 1962; Howard & Drazin 1964; Drazin, Beaumont & Coaker 1982; Maslowe 1991; Balmforth & Piccolo 2001) have presented inviscid linear stability calculations of the Bickley jet. For the unbounded Bickley jet parts of the stability boundary in the (β, k)-plane (where k is the wavenumber) are known analytically, as given by Lipps (1962), Howard & Drazin (1964) and Maslowe (1991). Two neutral modes, one sinuous and one varicose, and the corresponding neutral curves were found analytically by Lipps (1962); these modes are exponentially decaying at infinity. The analytic part of the stability boundary found by Maslowe (1991), which consists of sinuous modes that are bounded at infinity but neither radiate nor decay exponentially, comprises part of the stability boundary for the unstable radiating modes. Maslowe (1991) also found numerically that the neutral curve given by Howard & Drazin (1964) forms part of the stability boundary for the varicose mode, as opposed to the conclusion of Howard & Drazin which was that it was a sinuous mode and not a part of the stability boundary. The missing portions of the stability boundary for the sinuous mode of instability are computed numerically in the paper of Maslowe (1991). On the part which descends from the point $(\beta, k) = (-2, \sqrt{2})$ he found by a careful numerical study that the neutral modes must have the wave velocity $c = 1$. However, the stability boundary which is obtained differs somewhat from the correct one. In this note it is shown analytically that it is in fact the line $(\beta = -2,\ 0 < k < \sqrt{2})$ which bounds the instability region of the radiating sinuous modes to the left in the (β, k)-plane and that it consists of radiating sinuous modes with $c = 1$.

Concerning the bounded Bickley jet, no parts of the stability boundary and no neutral modes have previously been found analytically. However, in this case as well sinuous and varicose modes and corresponding neutral curves which comprise parts of the stability boundary can be found analytically, as shown in this note. Moreover, some local stability analysis are presented both for the sinuous and the varicose modes.

The plan of the paper is as follows. Section 2 introduces the basic equation and the Legendre equation governing the neutral solutions which are analytic at the critical layer, and the solutions of this particular equation are given. In §§ 3 and 4 analytical calculations for the bounded and the unbounded Bickley jet are presented.

2. Mathematical formulation

The linear stability of a parallel shear flow in the framework of the beta-plane approximation is considered. The basic flow velocity is assumed to be directed along the x-axis and varies in the vertical y-direction. The basic equation describing the evolution of the flow is the vorticity equation, which is linearized to give the equation governing the small-amplitude disturbances. If the perturbation stream function $\Psi = \phi(y) \exp\{ik(x - ct)\}$ is introduced into the linearized vorticity equation the Rayleigh–Kuo equation is obtained, i.e.

$$(U - c)(\phi'' - k^2\phi) - (U'' - \beta)\phi = 0, \tag{2.1}$$

where $U(y)$ is the basic flow velocity, $\phi(y)$ the amplitude function, k the wavenumber, c the wave velocity which may be complex, and β measures the Coriolis effect. The flow either may be bounded by two horizontal planes at $y = \pm L$ and then the boundary conditions are $\phi = 0$ at $y = \pm L$, or it may be of infinite extent and then ϕ is finite when $y \to \pm\infty$.

We investigate the barotropic instability of the Bickley jet where $U = \mathrm{sech}^2(y)$. Although β is always positive, changing its sign is mathematically equivalent to reversing the flow direction (Howard & Drazin 1964); therefore the results presented below for $\beta < 0$ and $U = \mathrm{sech}^2(y)$ correspond in reality to the retrograde jet $U = -\mathrm{sech}^2(y)$ with $\beta > 0$.

Introducing the new variable $\zeta = \tanh(y)$ into (2.1) with $U = \mathrm{sech}^2(y)$ yields

$$(1 - \zeta^2)\phi'' - 2\zeta\phi' + \left(6 - \frac{k^2}{1 - \zeta^2} - \frac{(4 - 6c)(1 - \zeta^2) - \beta}{(1 - \zeta^2)(1 - \zeta^2 - c)}\right)\phi = 0, \tag{2.2}$$

where the prime now denotes differentiation with respect to ζ.

If $\beta = (4 - 6c)c$ then (2.2) has no singularity at the critical layer where $U(y) = c$, and the equation becomes

$$(1-\zeta^2)\phi'' - 2\zeta\phi' + \left(6 - \frac{m^2}{1-\zeta^2}\right)\phi = 0, \quad \text{where} \quad m^2 = k^2 + 4 - 6c. \tag{2.3}$$

For $m = 0$ this is the Legendre equation with the solution $P_2(\zeta)$, the Legendre polynomial of degree two, which is finite at $\zeta = \pm 1$ ($\zeta = \pm 1$ corresponds to $y = \pm\infty$). This solution was given by Maslowe (1991) and the corresponding neutral curve comprises part of the stability boundary. For $m \neq 0$ this is the associated Legendre equation, which, when $m = 1$ and $m = 2$, has the solutions $P_2^1(\zeta)$ and $P_2^2(\zeta)$, the associated Legendre functions of the first kind, which are finite at $\zeta = \pm 1$. These solutions were given by Lipps (1962) and the corresponding neutral curves form parts of the stability boundary as well. However, it was noted by Engevik (2000) that (2.3) has the following simple closed-form solutions for all values of m:

$$\left.\begin{aligned} \phi_1 &= \left(\frac{1-\zeta}{1+\zeta}\right)^{m/2}(3\zeta^2 + 3m\zeta + m^2 - 1), \\ \phi_2 &= \left(\frac{1+\zeta}{1-\zeta}\right)^{m/2}(3\zeta^2 - 3m\zeta + m^2 - 1). \end{aligned}\right\} \tag{2.4}$$

These solutions are linearly independent when $m \neq 0, 1, 2$; for $m = 0, 1, 2$ they reduce to $P_2(\zeta)$, $P_2^1(\zeta)$ and $P_2^2(\zeta)$ respectively.

If we express these solutions in the variable y, they become

$$\left.\begin{aligned} \phi_1 &= \mathrm{e}^{-my}(3\tanh^2(y) + 3m\tanh(y) + m^2 - 1), \\ \phi_2 &= \mathrm{e}^{my}(3\tanh^2(y) - 3m\tanh(y) + m^2 - 1). \end{aligned}\right\} \tag{2.5}$$

In the context of the stability of the Bickley jet m^2 may be both positive and negative, i.e. m may be either real or purely imaginary which may give rise to either exponentially decaying modes or radiating modes at infinity.

3. Bounded Bickley jet

3.1. *Sinuous modes*

The neutral solutions for the sinuous modes are even functions of y and are obtained from (2.5). When m is real we find that

$$\left.\begin{aligned} \phi_0(y) &= \left(3\tanh^2(y) + \alpha_0^2 - 1\right)\cosh(\alpha_0 y) - 3\alpha_0\tanh(y)\sinh(\alpha_0 y), \\ c &= \tfrac{2}{3} - \tfrac{1}{6}\left(\alpha_0^2 - k^2\right), \end{aligned}\right\} \tag{3.1}$$

and the corresponding neutral curve

$$\beta = \tfrac{1}{6}\left(k^2 - \alpha_0^2 + 4\right)\left(\alpha_0^2 - k^2\right). \tag{3.2}$$

Here $\alpha_0 \neq 1$ is the solution of the equation

$$\tanh(mL) = \frac{3\tanh^2(L) + m^2 - 1}{3m\tanh(L)}, \tag{3.3}$$

when such a solution exists, which is the case when $3\tanh^2(L) - 1 \geqslant 0$, i.e. $L \geqslant L_0 = 0.65847\cdots$. When $L < L_0$ no sinuous mode of the kind given in (3.1) exists. However, (3.3) has the solution $m = 1$ for all values of L as well, but the amplitude function

$\phi(y)$ is equal to zero when $m = 1$, so we have no sinuous mode for this value of m. When $L \to \infty$ then $\alpha_0 \to 2$ and the sinuous mode given by Lipps (1962) is obtained.

When m is purely imaginary a number of neutral modes and corresponding neutral curves may exist; the number increases with L. We find that

$$\left.\begin{aligned} \phi_j &= \left(3\tanh^2(y) - \alpha_j^2 - 1\right)\cos(\alpha_j y) + 3\alpha_j \tanh(y)\sin(\alpha_j y), \\ c &= \tfrac{2}{3} + \tfrac{1}{6}\left(\alpha_j^2 + k^2\right), \end{aligned}\right\} \quad j = 1, \dots, N. \tag{3.4}$$

The corresponding neutral curves are

$$\beta = -\tfrac{1}{6}\left(k^2 + \alpha_j^2 + 4\right)\left(k^2 + \alpha_j^2\right). \tag{3.5}$$

Here α_j is the solution of the equation

$$\tan(\alpha L) = -\frac{3\tanh^2(L) - \alpha^2 - 1}{3\alpha \tanh(L)} \tag{3.6}$$

and is in addition subjected to the condition $\alpha_j < \sqrt{2}$ since $\beta > -2$. (A necessary condition for instability is that $-2 < \beta < 2/3$, which follows from a generalization of Rayleigh's inflection point theorem giving that the quantity $(U'' - \beta)$ must change sign within the flow region for instability to occur.) There are only a finite number of solutions of (3.6) which satisfy this condition. (When $L < L_0$ no $\alpha_j < \sqrt{2}$ exists, so then there are no sinuous modes of either the kind given in (3.1) or of the kind given in (3.4).) We notice that all of these neutral curves are parts of parabolas in the (β, k^2)-plane, those parts for which $-2 \leqslant \beta \leqslant 2/3$ and $k^2 \geqslant 0$.

To determine on what side of the neutral curve in the (β, k)-plane there is instability we have to know the stability characteristics in the neighbourhood of the curve, which can be achieved by a perturbation of the known neutral solution. Let the amplitude function, the wave velocity, the wavenumber and the beta-parameter for the neutral solution and the contiguous unstable solution be denoted by ϕ_s, c_s, k_s, β_s, ϕ, c, k and β respectively. Both the unstable and the neutral solution satisfy (2.1) and the boundary conditions, from which it follows that

$$\begin{aligned} &(c - c_s)\int_{-L}^{L} \frac{(6\tanh^2(y) - 2 - 6c_s)\phi(y)\phi_s(y)}{\tanh^2(y) - (1 - c)}\,\mathrm{d}y \\ &\quad - \left(k^2 - k_s^2\right)\int_{-L}^{L} \phi(y)\phi_s(y)\,\mathrm{d}y - (\beta - \beta_s)\int_{-L}^{L} \frac{\phi(y)\phi_s(y)}{\tanh^2(y) - (1 - c)}\,\mathrm{d}y = 0. \end{aligned} \tag{3.7}$$

Let $\beta = \beta_s$. When $k \to k_s$, then $c \to c_s$ and $\phi \to \phi_s$ and applying Plemelj's formula (see Muskhelishvili 1953) it follows from (3.7) that

$$c - c_s = \frac{(I_1 + \mathrm{i}\pi K)I_0\left(k^2 - k_s^2\right)}{I_1^2 + \pi^2 K^2}, \tag{3.8}$$

where

$$\left.\begin{aligned} K &= \frac{(2 - 6c_s)\phi_s^2(y_s)}{c_s\sqrt{1 - c_s}}, \\ I_0 &= \int_0^L \phi_s^2(y)\,\mathrm{d}y, \\ I_1 &= P\int_0^L \frac{(6\tanh^2(y) - 2 - 6c_s)\phi_s^2(y)}{\tanh^2(y) - (1 - c_s)}\,\mathrm{d}y, \end{aligned}\right\} \tag{3.9}$$

where $\phi_s(y)$ is given by either (3.1) or (3.4), y_s is the solution of the equation $\tanh(y)=\sqrt{1-c_s}$, and P in front of the integral sign denotes the principal value of the integral. From (3.8) it follows that when $K<0$ there is instability for $k<k_s$ and when $K>0$ instability occurs for $k>k_s$. This means that there is instability below that part of the neutral curve in (3.2) where $k>\sqrt{\alpha_0^2-2}$ and above that part where $k<\sqrt{\alpha_0^2-2}$. For the modes given by (3.4) $K<0$ for all j so there is instability below these neutral curves.

K in (3.9) is not defined for $c_s=1$, which occurs at the points where the neutral curves intersect the line $\beta=-2$. However, (3.7) is still valid with $c_s=1$, $\beta_s=-2$, and $k_s=\hat{k}$, where $\hat{k}$ is equal to either $\sqrt{2+\alpha_0^2}$ or $\sqrt{2-\alpha_j^2}$, $j=1,\ldots,N$. We put $c=1-\bar{c}$, $\beta=-2+\bar{\beta}$ and $\kappa=k-\hat{k}$, where $|\bar{c}|\ll 1$, $\bar{\beta}\ll 1$ and $|\kappa|\ll 1$, into (3.7) and obtain, when only the dominant terms are retained,

$$\bar{c}-\mathrm{i}\Gamma\kappa\sqrt{\bar{c}}-\frac{\bar{\beta}}{8}=0, \tag{3.10}$$

where $\Gamma=I_0\hat{k}/2\pi\phi_s^2(0)$, I_0 is given by (3.9) and $\phi_s^2(0)$ by either (3.1) or (3.4). Equation (3.10) reveals instability if $\kappa<0$ and $\bar{\beta}>2\Gamma^2\kappa^2$ and then

$$\bar{c}=\tfrac{1}{4}(R^2-\Gamma^2\kappa^2+2\mathrm{i}\Gamma R\kappa),\ \text{where}\ R=\sqrt{\tfrac{1}{2}\bar{\beta}-\Gamma^2\kappa^2}. \tag{3.11}$$

Let $(\beta,k)=(-2,\hat{k})$ be the point of intersection between either one of the neutral curves given by (3.2) or (3.5) and the line $\beta=-2$. It follows from the analysis above that near this point there is instability in a wedge-like region bounded above by this neutral curve and below by the curve $\bar{\beta}=2\Gamma^2\kappa^2$, which is a parabola in the (β,k)-plane. In this instability region $\bar{c}_r<0$ when $2\Gamma^2\kappa^2<\bar{\beta}<4\Gamma^2\kappa^2$, and $\bar{c}_r>0$ when $\bar{\beta}>4\Gamma^2\kappa^2$. When $\bar{\beta}=4\Gamma^2\kappa^2$ then $\bar{c}$ is purely imaginary. In non-rotational shear flows Howard's (1961) semicircle theorem states that the real part of the complex wave velocity $c=c_r+\mathrm{i}c_i$ must lie within the range of the velocity profile $U(y)$. However, a modified version of this semicircle theorem due to Pedlosky (1964) allows for c_r to be outside the range of $U(y)$ if the beta-effect is included, which is what happens in the instability region where $\bar{c}_r<0$.

Let $L=10$. To obtain the neutral curves given by (3.2) and (3.5) we have to solve (3.3) and (3.6) to find α_0 and the α_j, $j=1,2,\ldots$ for which $\alpha_j<\sqrt{2}$. We find that $\alpha_0=2.00$, $\alpha_1=0.18$, $\alpha_2=0.55$, $\alpha_3=0.90$, $\alpha_4=1.24$, and the corresponding neutral curves which are labelled (*a*), (*b*), (*c*), (*d*) and (*e*) are shown in figure 1. It is found that the solution of (3.3) differs very little from 2 so the curve labelled (*a*) is in fact the neutral curve for the sinuous modes found by Lipps (1962) for the unbounded Bickley jet. This illustrates that the neutral mode corresponding to α_0, which is decaying exponentially outward, is not very sensitive to the boundary conditions. The stability analysis above shows that these neutral curves form parts of the stability boundary. These curves have been computed numerically by Balmforth & Piccolo (2001). Futhermore, we have shown analytically that near the points where the neutral curves (*a*), (*b*), (*c*), (*d*) and (*e*) intersect the line $\beta=-2$ there is instability in wedge-like regions bounded above by these neutral curves and below by curves that are parabolas on which the neutral modes have $c_r>1$ in agreement with the numerical findings of Balmforth & Piccolo (2001). While the upper boundary of each tongue in figure 1 can be found analytically this is not so for the lower boundary except in a region close to $\beta=-2$. On the lower boundary it appears to be impossible to

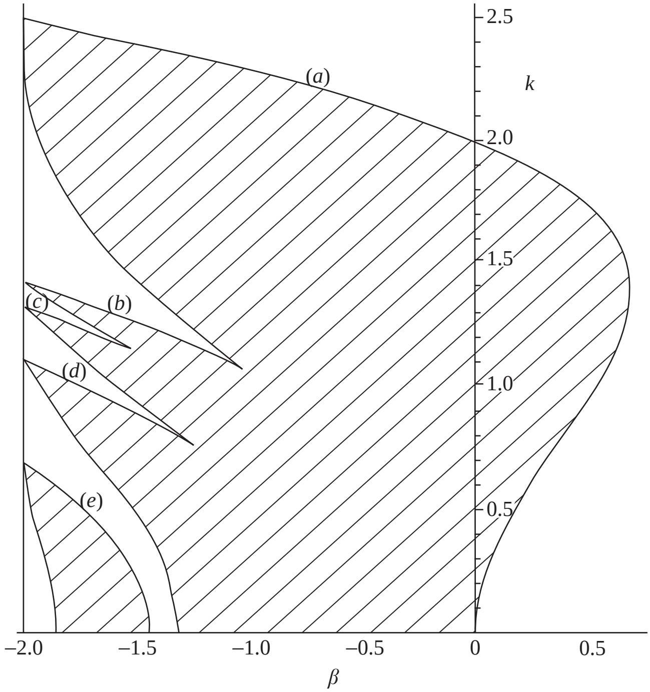

FIGURE 1. The hatched regions are the instability regions for the sinuous modes when $L = 10$. The upper boundaries of the tongues are the neutral curves: $\beta = (k^2 - \alpha_0^2 + 4)(\alpha_0^2 - k^2)/6$ and $\beta = -(k^2 + \alpha_j^2 + 4)(\alpha_j^2 + k^2)/6$, $j = 1, 2, 3, 4$, labelled (a), (b), (c), (d) and (e) respectively, where $\alpha_0 = 2.00$, $\alpha_1 = 0.18$, $\alpha_2 = 0.55$, $\alpha_3 = 0.90$ and $\alpha_4 = 1.24$. The lower boundaries of the tongues have been computed numerically by Balmforth & Piccolo (2001).

transform the equation that governs the neutral modes into any well-known equation with simple closed-form solutions, which is the case on the upper boundary.

3.2. *Varicose modes*

The neutral solutions for the varicose modes are odd functions of y and are obtained from (2.5) as well. In this case the neutral solutions, the wave velocities and the neutral curves are as follows:

$$\left.\begin{aligned} \phi_0(y) &= \left(3\tanh^2(y) + \alpha_0^2 - 1\right)\sinh(\alpha_0 y) - 3\alpha_0\tanh(y)\cosh(\alpha_0 y),\\ c &= \tfrac{2}{3} - \tfrac{1}{6}\left(\alpha_0^2 - k^2\right),\\ \beta &= \tfrac{1}{6}\left(k^2 - \alpha_0^2 + 4\right)\left(\alpha_0^2 - k^2\right), \end{aligned}\right\} \tag{3.12}$$

and

$$\left.\begin{aligned} \phi_j &= \left(3\tanh^2(y) - \alpha_j^2 - 1\right)\sin(\alpha_j y) - 3\alpha_j\tanh(y)\cos(\alpha_j y),\\ c &= \tfrac{2}{3} + \tfrac{1}{6}\left(\alpha_j^2 + k^2\right),\\ \beta &= -\tfrac{1}{6}\left(k^2 + \alpha_j^2 + 4\right)\left(k^2 + \alpha_j^2\right). \end{aligned}\right\} \quad j = 1, \cdots, N. \tag{3.13}$$

Here $\alpha_0 \neq 2$ is the solution of the equation

$$\coth(mL) = \frac{3\tanh^2(L) + m^2 - 1}{3m\tanh(L)}, \tag{3.14}$$

when such a solution exists, i.e. when $L \geqslant L_0$. However, (3.14) has the solution $m = 2$ for all values of L as well, but the amplitude function $\phi(y)$ is equal to zero when $m = 2$, so we have no varicose mode for this value of m. When $L \to \infty$ then $\alpha_0 \to 1$ and the varicose mode given by Lipps (1962) is obtained.

We now find α_j, $j = 1, \ldots, N$, which is the solution of the equation

$$\cot(\alpha L) = \frac{3\tanh^2(L) - \alpha^2 - 1}{3\alpha\tanh(L)}, \tag{3.15}$$

where $\alpha_j < \sqrt{2}$ since $\beta > -2$. (When $L < L_0$ there is no $\alpha_j < \sqrt{2}$, and then there are no varicose modes, either of the kind given in (3.12) or of the kind given in (3.13).)

The instability side of these neutral curves is obtained from (3.8), which is applicable in this case as well. We find that there is instability on the lower side of these neutral curves. Near the points where the neutral curves intersect the line $\beta = -2$ the instability region is determined from the following equation, where only the dominant terms have been retained:

$$\{I_2 + 4\mathrm{i}\pi A\sqrt{\bar{c}_r}\}\bar{c} + 2\hat{k}I_0\kappa + \{I_3 - \tfrac{1}{2}\mathrm{i}\pi A\sqrt{\bar{c}_r}\}\bar{\beta} = 0, \tag{3.16}$$

where $\bar{c}$, κ, $\bar{\beta}$, $\hat{k}$ and I_0 are defined previously and

$$\left.\begin{aligned} I_2 &= \int_0^L \frac{(6\tanh^2(y) - 8)\phi_s^2(y)}{\tanh^2(y)}\,\mathrm{d}y, \qquad I_3 = \int_0^L \frac{\phi_s^2(y)}{\tanh^2(y)}\,\mathrm{d}y, \\ A &= \left(\alpha_0^3 - 4\alpha_0\right)^2 \quad \text{or} \quad \left(\alpha_j^3 + 4\alpha_j\right)^2, \ j = 1, \cdots, N, \end{aligned}\right\} \tag{3.17}$$

where we notice that $I_2 < 0$.

From (3.16) it follows that

$$\bar{c}_r = -\frac{1}{I_2}\{2\hat{k}I_0\kappa + I_3\bar{\beta}\}, \qquad \bar{c}_i = \frac{\pi A\sqrt{\bar{c}_r}}{2I_2}\{\bar{\beta} - 8\bar{c}_r\}. \tag{3.18}$$

There is instability when $\bar{c}_i < 0$ which corresponds to $c_i > 0$. This occurs in a wedge bounded above by either of the neutral curves in (3.12) or (3.13) and below by the curve $2\hat{k}I_0\kappa + I_3\bar{\beta} = 0$, which follows from (3.18). Notice that $\bar{\beta} - 8\bar{c}_r = 0$ to the lowest order on the neutral curves given by (3.12) or (3.13) near the points where these curves intersect the line $\beta = -2$. In the instability region $\bar{c}_r > 0$, which means that in this case c_r lies within the range of $U(y)$, as opposed to what was found in the sinuous modes case where unstable modes with c_r outside the range of $U(y)$ occur.

The neutral curves for the varicose modes have also been obtained in the case when $L = 10$. It is found that $\alpha_0 = 0.99999998$, $\alpha_1 = 0.37$, $\alpha_2 = 0.73$, $\alpha_3 = 1.07$, $\alpha_4 = 1.4137$, and the corresponding neutral curves labelled (*a*), (*b*), (*c*) and (*d*) are shown in figure 2, except for the curve corresponding to α_4 which is so close to the lower left corner of the diagram that it does not show in the figure. However, there is a small instability region here which has not been noted previously. We see that α_0 is almost equal to 1 when $L = 10$ so the curve labelled (*a*) is very close the neutral curve for the varicose modes found by Lipps (1962), and this shows that the mode corresponding to α_0 is not very sensitive to a change in L. These neutral curves comprise parts of the stability boundary, which follows from the stability analysis above. Figure 2 shows four separate instability regions, each being bounded above by these neutral curves. The upper instability region is bounded below by the curve $\beta = -k^2(1 - k^2/9)$ (Howard & Drazin 1964), which consists of neutral modes with $c_r = 1$. This curve was found by Howard & Drazin for the unbounded Bickley jet,

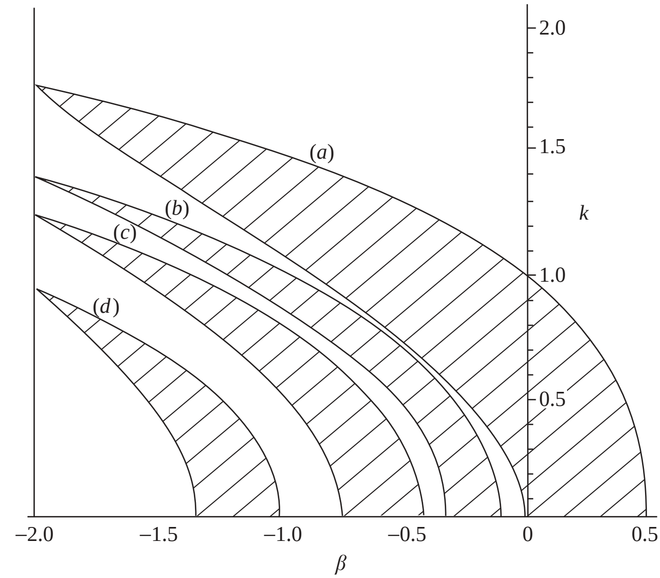

FIGURE 2. The hatched regions are the instability regions for the varicose modes when $L = 10$. They are bounded above by the neutral curves: $\beta = (k^2 - \alpha_0^2 + 4)(\alpha_0^2 - k^2)/6$ and $\beta = -(k^2 + \alpha_j^2 + 4)(\alpha_j^2 + k^2)/6$, $j = 1, 2, 3$, labelled (a), (b), (c) and (d) respectively, where $\alpha_0 = 1.00$, $\alpha_1 = 0.37$, $\alpha_2 = 0.73$, $\alpha_3 = 1.07$. The upper instability region is bounded below by the neutral curve $\beta = -k^2(1 - k^2/9)$ (Howard & Drazin 1964). The lower boundaries of the other regions have been computed numerically by Balmforth & Piccolo (2001).

but it seems to be a good approximation when $L = 10$. The curves that yield the lower boundaries for the other instability regions have been computed numerically by Balmforth & Piccolo (2001) who found that these boundaries consist of neutral modes with $c_r = 1$ as well.

In this note analytic expressions for these curves, valid near $\beta = -2$, are found. The stability analysis above shows instability in wedge-shaped regions near $\beta = -2$, the lower boundaries of which are lines where the neutral modes have $c_r = 1$. The upper boundaries of the instability regions in figure 2, which are found analytically in this note, have also been computed numerically by Balmforth & Piccolo.

4. Unbounded Bickley jet

The hatched region in figure 3 shows the instability region for the sinuous modes. It is bounded by the neutral curves labelled (a), (b), (c) and (d). Three of them, labelled (a), (c) and (d) respectively, are given analytically: $\beta = k^2(4 - k^2)/6$ (Lipps 1962), $\beta = -k^2(k^2 + 4)/6$ (Maslowe 1991) and $\beta = -2$, $0 < k < \sqrt{2}$ (given in this note). The curves labelled (a) and (c) and the corresponding neutral modes can be obtained from the results in § 2 if m is taken to be 2 and 0 respectively. The curve labelled (b) has been computed numerically by Maslowe (1991). Maslowe reported that numerical results for $c_i > 0$ obtained by Deblonde (1981) show that weakly amplified modes exist that do not have critical layers when $c_i \to 0^+$ in the region towards the upper left of the stability diagram in figure 3. This is in agreement with the analytical results given in this note. The stability characteristics in this region, which is bounded below by the curve $\bar{\beta} = 4\kappa^2/3\pi^2$ locally, are governed by (3.10) with $\Gamma = \sqrt{6}/3\pi$ since

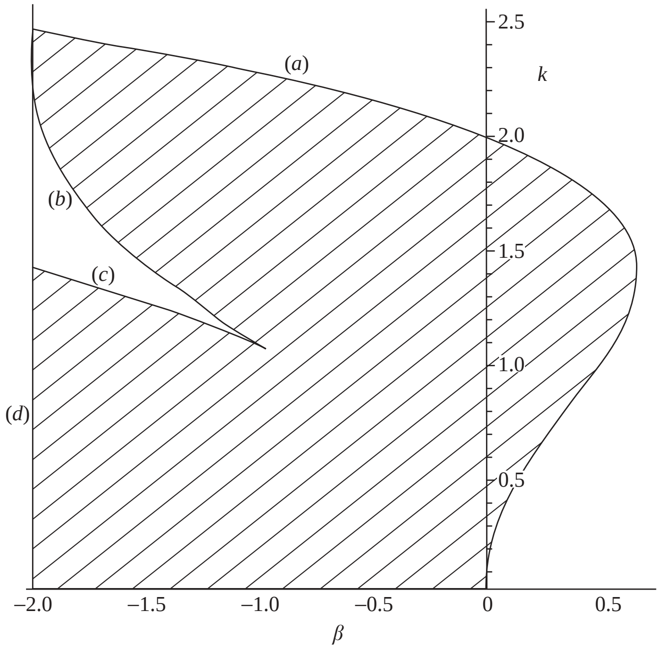

FIGURE 3. The hatched region is the instability region for the sinuous modes of the unbounded Bickley jet. The instability region is bounded by the curves: $\beta = k^2(4-k^2)/6$ (Lipps 1962), $\beta = -k^2(k^2+4)/6$ (Maslowe 1991) and $\beta = -2,\ 0 < k < \sqrt{2}$ labelled (*a*), (*c*) and (*d*) respectively, and the curve labelled (*b*) has been computed numerically by Maslowe (1991).

$\phi_s = \mathrm{sech}^2(y)$ in this case, and we find that

$$\bar{c} = \frac{1}{4}\left\{R^2 - \frac{2\kappa^2}{3\pi^2} + \frac{2\sqrt{6}R\kappa}{3\pi}\,\mathrm{i}\right\}, \qquad R = \sqrt{\frac{\bar{\beta}}{2} - \frac{2\kappa^2}{3\pi^2}}. \tag{4.1}$$

We see that $\bar{c}_r$ changes sign in this instability region; $\bar{c}_r < 0$ where $4\kappa^2/3\pi^2 < \bar{\beta} < 8\kappa^2/3\pi^2$ which means that $c_r > 1$ and therefore lies outside the range of $U = \mathrm{sech}^2(y)$ in accordance with the numerical findings of Deblonde as mentioned above. $\bar{c}$ is purely imaginary along the curve $\bar{\beta} = 8\kappa^2/3\pi^2$, and in the rest of the instability region $\bar{c}_r > 0$ which means that here c_r lies within the range of $U(y)$.

Figure 3 corresponds to figure 2 in the paper of Maslowe (1991) except for that portion of the stability boundary which descends from the point $(\beta, k) = (-2, \sqrt{2})$. After a careful numerical study Maslowe concluded that this part of the stability boundary must consist of singular radiating modes with $c = 1$. He gave a numerical method to calculate this part of the stability boundary which is shown in figure 2 of his paper. However, the stability boundary is in fact the line ($\beta = -2$, $k = \sqrt{2-\alpha^2}$, where $0 < \alpha < \sqrt{2}$), and this part consists of radiating modes with $c = 1$; the amplitude function being

$$\phi_s(y) = \begin{cases} \phi_1 = \mathrm{e}^{-\mathrm{i}\alpha y}(3\tanh^2(y) + 3\mathrm{i}\alpha\tanh(y) - \alpha^2 - 1) & \text{for } y > 0, \\ \phi_2 = \mathrm{e}^{\mathrm{i}\alpha y}(3\tanh^2(y) - 3\mathrm{i}\alpha\tanh(y) - \alpha^2 - 1) & \text{for } y < 0. \end{cases} \tag{4.2}$$

This follows from the results in § 2. We see that $\phi_s(y)$ satisfies the radiation condition when $y \to \pm\infty$ as given by Maslowe (1991) and that there is a jump in the Reynolds stress across the critical layer $y = 0$.

Contiguous to the neutral solution an unstable solution exists with the wave velocity $c = 1 - \bar{c}$, the wavenumber $k = k_s$ and the parameter $\beta = -2 + \bar{\beta}$, where $|\bar{c}| \ll 1$ and $\bar{\beta} \ll 1$, and we find that

$$\bar{c} = \frac{(1+\alpha^2)^2\pi^2}{4\alpha^2(\alpha^2+4)^2}\bar{\beta}^2 - \mathrm{i}\Lambda\bar{\beta}^3 + \cdots, \tag{4.3}$$

where Λ is positive and depends on α. For details, see the Appendix. As we see this is a very weak instability.

5. Conclusions

The equation governing the neutral solutions which are analytic at the critical layer can be transformed into an associated Legendre equation, and this particular equation has simple closed-form solutions which have been overlooked in previous works on barotropic stability of the Bickley jet. Applying these solutions leads to new analytic neutral modes and corresponding neutral curves which form parts of the stability boundary in the (β, k)-plane. For the unbounded Bickley jet this yields that the line segment $(\beta = -2,\ 0 < k < \sqrt{2})$ is a portion of the stability boundary and that it consists of sinuous modes with wave velocity $c = 1$. Therefore, the instability region of the radiating sinuous modes is bounded to the left in the (β, k)-plane by this line segment and is larger than the one given previously. For the bounded Bickley jet, for which case only numerical calculations have been presented so far, this leads to analytic sinuous and varicose modes and neutral curves that comprise parts of the stability boundary; the number of these modes increases with the distance between the planes which bound the flow.

An equation which yields the stability characteristics for the sinuous modes near the point $(\beta = -2, k = \sqrt{6})$ on the stability boundary of the unbounded Bickley jet is obtained. It reveals weakly unstable modes which have wave speed outside the range of the velocity profile $U(y)$ as well as modes with wave speed inside this range. The former are rather rare but have been revealed previously by numerical calculations. In the case of the bounded Bickley jet a number of neutral sinuous modes exists which intersect the line $\beta = -2$, and the stability characteristics near these points are given by the same type of equation as the one found for the unbounded flow, so in this case as well weakly unstable modes exist with wave speed outside the range of $U(y)$. An equation valid in the instability region near the points of intersection of the neutral varicose modes and the line $\beta = -2$ is also obtained, but this equation reveals no weakly unstable modes with wave speed outside the range of $U(y)$.

Appendix

We consider a solution ϕ close to ϕ_s given by (4.2) with the corresponding wave velocity $c = 1 - \bar{c}$, the wavenumber $k = k_s$ and the parameter $\beta = -2 + \bar{\beta}$, where $|c| \ll 1$ and $\bar{\beta} \ll 1$. The equation for $\phi(y)$ is obtained from (2.1) with $U = \mathrm{sech}^2(y)$, which yields

$$\phi'' + (6\mathrm{sech}^2(y) + \bar{\alpha}^2)\phi - F(U, \bar{\beta}, \bar{c})\phi = 0, \tag{A 1}$$

where

$$\left.\begin{aligned} &\bar{\alpha}^2 = \alpha^2 + \frac{2\bar{c} - \bar{\beta}}{1 - \bar{c}}, \\ &F(U, \bar{\beta}, \bar{c}) = \frac{\gamma}{\cosh^2(y)(\tanh^2(y) - \bar{c})}, \quad \gamma = \frac{\bar{\beta} - 8\bar{c} + 6\bar{c}^2}{1 - \bar{c}}. \end{aligned}\right\} \tag{A 2}$$

The equation

$$\phi'' + (6\operatorname{sech}^2(y) + \bar{\alpha}^2)\phi = 0 \tag{A 3}$$

has two linearly independent solutions when $\bar{\alpha} \neq 0$,

$$\left.\begin{aligned} \psi_{0+} &= e^{-i\bar{\alpha}y}(3\tanh^2(y) + 3i\bar{\alpha}\tanh(y) - \bar{\alpha}^2 - 1), \\ \psi_{0-} &= e^{i\bar{\alpha}y}(3\tanh^2(y) - 3i\bar{\alpha}\tanh(y) - \bar{\alpha}^2 - 1). \end{aligned}\right\} \tag{A 4}$$

We see that ψ_{0+} and ψ_{0-} are equal to ϕ_1 and ϕ_2 given by (4.2) respectively if $\bar{\alpha}$ is replaced by α.

Since $\bar{\beta} \ll 1$ and $|\bar{c}| \ll 1$ two approximate solutions ϕ_+ and ϕ_- to (A 1) can be written as

$$\phi_\pm = \psi_{0\pm} + \psi_{1\pm} + \psi_{2\pm}, \tag{A 5}$$

where

$$\left.\begin{aligned} \psi_{1\pm} &= \frac{\psi_{0\pm}}{W_\pm}\int_y^{\pm\infty} F\psi_{0+}\psi_{0-}\,d\tau - \frac{\psi_{0\mp}}{W_\pm}\int_y^{\pm\infty} F\psi_{0\pm}^2\,d\tau, \\ \psi_{2\pm} &= \frac{\psi_{0\pm}}{W_\pm}\int_y^{\pm\infty} F\psi_{1\pm}\psi_{0\mp}\,d\tau - \frac{\psi_{0\mp}}{W_\pm}\int_y^{\pm\infty} F\psi_{1\pm}\psi_{0\pm}\,d\tau, \end{aligned}\right\} \tag{A 6}$$

where $W_\pm = \psi_{0\pm}\psi_{0\mp}{}' - \psi_{0\pm}{}'\psi_{0\mp}$; ϕ_+ satisfies the boundary condition at $y = +\infty$ while ϕ_- satisfies the condition at $y = -\infty$. In the region where they overlap their Wronskian must be zero if they are to be linearly dependent, i.e.

$$\phi_+(0)\phi_-{}'(0) - \phi_+{}'(0)\phi_-(0) = 0. \tag{A 7}$$

Introducing the expressions for ϕ_+ and ϕ_- into (A 7) yields

$$2i\bar{\alpha}(\bar{\alpha}^2 + 1)(\bar{\alpha}^2 + 4) + \int_0^\infty F\psi_{0+}\psi_{0-}\,d\tau + \int_0^\infty F\psi_{0+}^2\,d\tau + \int_0^\infty F\psi_{1+}\psi_{0+}\,d\tau + \int_0^\infty F\psi_{1+}\psi_{0-}\,d\tau = 0. \tag{A 8}$$

We expand the left-hand side of (A 8) in powers of $\bar{c}$ and $\bar{\beta}$, and the equation is satisfied to the lowest order if

$$\bar{c} = \frac{(1+\alpha^2)^2\pi^2}{4\alpha^2(\alpha^2+4)^2}\bar{\beta}^2 - i\Lambda\bar{\beta}^3 + \cdots. \tag{A 9}$$

This is the expression given in (4.3).

REFERENCES

Balmforth, N. J. & Piccolo, C. 2001 The onset of meandering in a barotropic jet. *J. Fluid Mech.* **449**, 84–114.

Deblonde, G. 1981 Instabilité barotrope du jet de Bickley. MSc thesis, McGill University.

Drazin, P. G., Beaumount, D. N. & Coaker, S. A. 1982 On Rossby waves modified by basic shear and barotropic instability. *J. Fluid Mech.* **124**, 439–456.

Engevik, L. 2000 A note on the instabilities of a horizontal shear flow with a free surface. *J. Fluid Mech.* **406**, 337–346.

Howard, L. N. 1961 Note on a paper of John W. Miles. *J. Fluid Mech.* **10**, 509–512.

Howard, L. N. & Drazin, P. G. 1964 On instability of parallel flow of inviscid fluid in a rotating system with variable Coriolis parameter. *J. Maths and Phys.* **43**, 83–99.

Lipps, F. B. 1962 The barotropic stability of the mean winds in the atmosphere. *J. Fluid Mech.* **12**, 397–407.

Maslowe, S. A. 1991 Barotropic instability of the Bickley jet. *J. Fluid Mech.* **229**, 417–426.

Muskhelishvili, N. I. 1953 *Singular Integral Equations.* Noordhoff.

Pedlosky, J. 1964 The stability of currents in the atmosphere and ocean: Part 1. *J. Atmos. Sci.* **21**, 201–219.